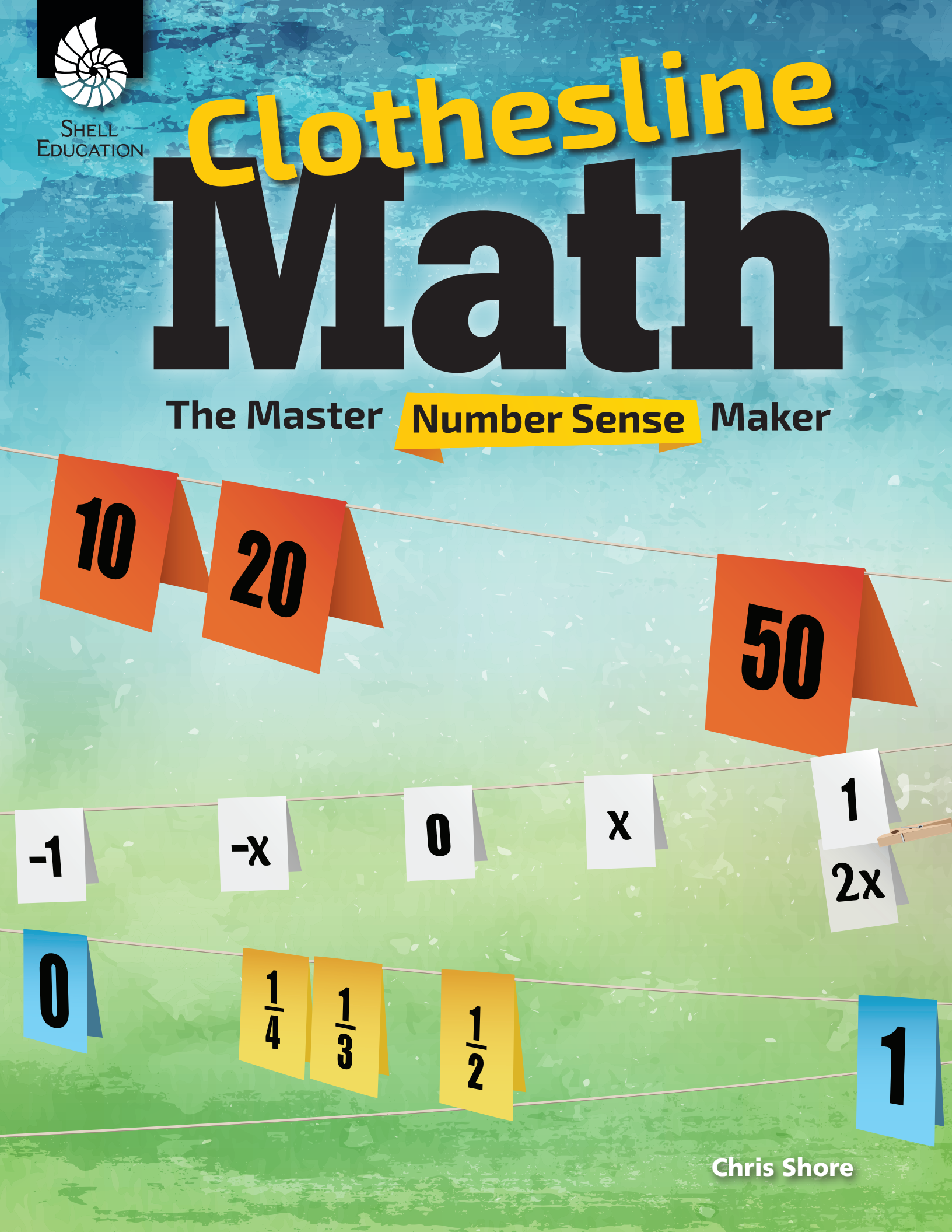
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The Master Number Sense Maker
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Chris Shore

Clothesline Math

The Master Number Sense Maker

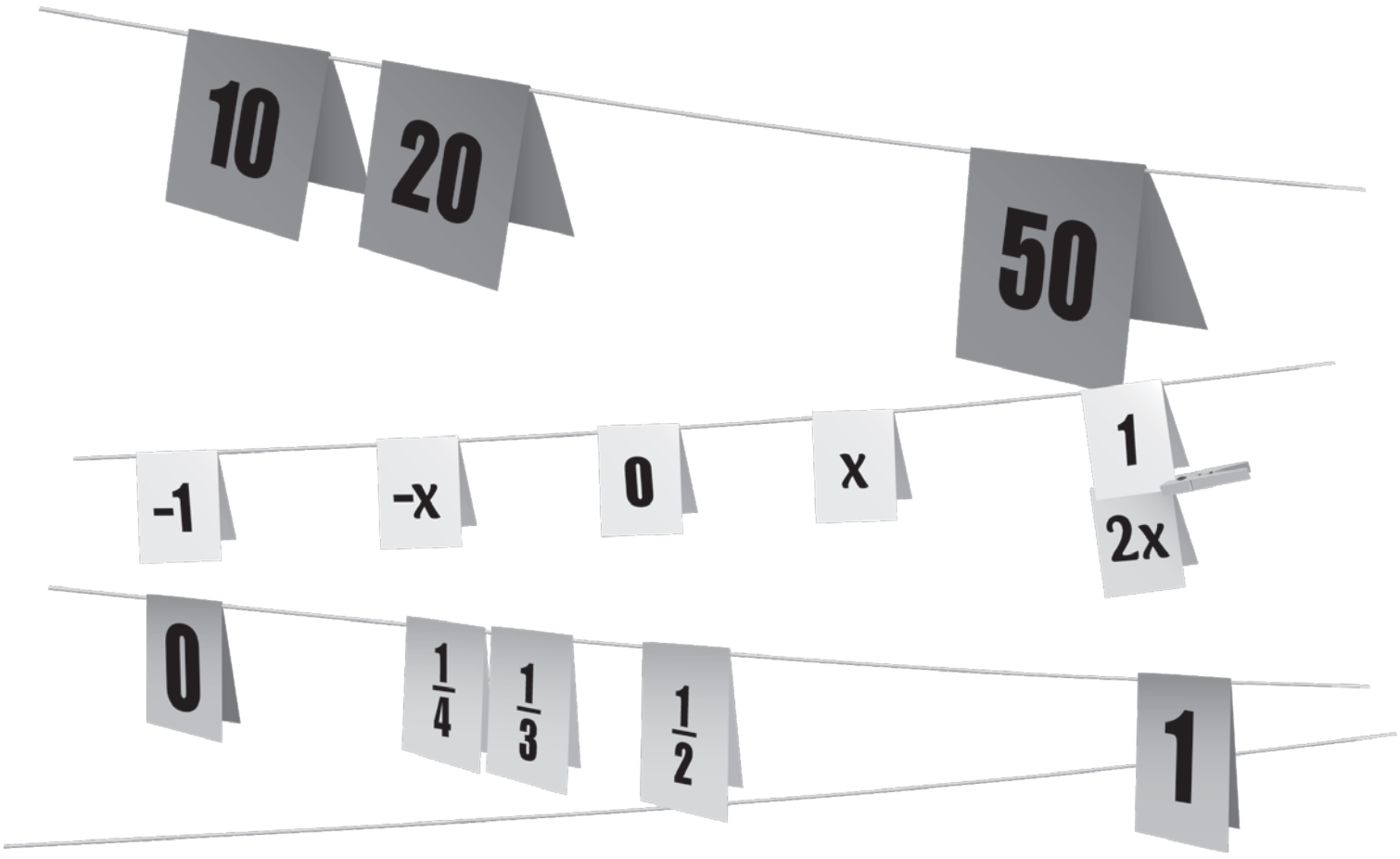

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Table of Contents

Imagine almost missing an international flight because you and a friend were engrossed in something that was rockin' your world. In October 2015, Chris Shore and I were those two friends, and we almost missed our flight to Vancouver. Picture two grown men surrounded by hundreds of strangers inside Los Angeles International Airport. We were excitedly talking and deeply focused on my math notebook as we used an open number line to solve a multistep math equation with variables on both sides. At one point, we had this magical eureka moment, and we celebrated our collaborative findings with a vigorous chest bump and extremely loud, "Oh yeah!"

Rest assured, we made our flight. More importantly, thanks to the open number line that day, our number sense received a serious boost. *Clothesline Math: The Master Number Sense Maker* by Chris Shore is your ticket for embarking on a similar journey that will boost your number sense and the number sense of your students.

Two months before our airport eureka chest bump, Chris came to Tustin Unified School District (TUSD) and provided secondary math teachers and instructional coaches with professional development. As a math instructional coach at TUSD, I was beyond excited to learn from him. He introduced TUSD to the open dynamic number line known as Clothesline Math. Shore immediately blew our math minds and challenged our number sense with lessons ranging from numerical relationships of fractions to mean absolute deviation. I immediately thought, "This is awesome! I wish I had this in my classroom when teaching middle-school math!" Looking across the room at all the other math teachers, I could see their faces filled with both excitement and apprehension. On one hand, teachers were thrilled to receive this amazing and transformative math gift. On the other hand, whenever teachers explore something so radical and dynamic like Clothesline Math, they desire and need a roadmap to guide their teaching and student learning. *Clothesline Math: The Master Number Sense Maker* is your roadmap to classroom-tested lessons and teacher moves for successfully facilitating Clothesline Math in your classroom.

This book begins with Chris doing a wonderful job citing research and best practices to explain why it is so valuable for teachers to provide their students with opportunities to strengthen their number sense. Number sense has a direct impact on the mathematical content students are expected to learn each year of their schooling. As Shore points out, "Students don't flunk current content; they flunk prior content." He then goes on to explain why teaching number sense and prerequisite skills are essential for students to be more successful in mathematics at their current grade level. An open number line, such as the Clothesline, is both a vehicle and a destination for students and teachers to boost their number sense and prerequisite skills.

After quickly convincing us why number sense lessons need to be a frequent part of the teaching and learning of mathematics, Chris takes us on a ride through various K–12 Clothesline Math lessons, such as numerical relationships, rational numbers, algebraic expressions, logarithms, and trigonometric relationships. The lessons are not "just one more thing" you should be doing. The lessons and resources in this book are framed in a way to enhance the content you're supposed to teach while empowering you to strengthen your craft of teaching mathematics at the same time.

I've seen how Chris has continued to both gather and share Clothesline Math wisdom and strategies with teachers around the country at conferences and other professional development. Most of the wisdom and strategies Chris shares in the book come from his own classroom experience as a math teacher. If you have ever met and had a conversation with him, you know Chris has the gift of intently listening and remembering every detail of conversations. His gift of listening for details shows in this book, as he has meticulously captured student and teacher conversations, showcasing them through vignettes. The vignettes capture the essence of a Clothesline lesson, benefiting teachers who are either new to Clothesline Math or are already familiar with its transformative powers. These vignettes model the necessary and essential mathematical discourse that go along with any Clothesline Math lesson, providing the roadmap teachers desire. Chris says it best: "Clothesline Math is intended for discourse, not lecture."

Chris has tested and experienced these lessons with students; he has listened intently to their comments, questions, and ideas. As you'll discover, Chris uses the perspective and voice of students to demonstrate how powerful and accessible Clothesline Math lessons are for all types of math students. Through these conversations, you'll quickly see that the clothesline is a student-centered tool for building number sense. It's our job as educators to identify when we need to get into the passenger seat and let students drive the learning. But don't worry—this book is still your roadmap as it provides card sets, organizational tips, conversations, and key questions for each lesson to help guide the conversations and learning that will transpire in your classroom.

Before you know it, you will find yourself engrossed in *Clothesline Math: The Master Number Sense Maker*, exploring the numerical relationships you teach and skills students need to be successful in learning mathematics. I highly encourage you to find a math friend and explore Clothesline Math lessons together. Get immersed in your notebooks, celebrate your findings (maybe with a chest bump), and, as Chris Shore would say, "Go change the world, one math lesson at a time." Just don't miss your flight.

—Andrew Stadel
Math Teacher, Instructional Coach, and Consultant

Thank you to…

Kelli Wise, for introducing me to the potential of Clothesline Math. That one session of yours at the MaTHink conference changed my teaching profoundly. This book would never have been written without your initial inspiration.

Tim McCaffery, for sharing that original Clothesline experience and giving the Clothesline its nickname, The Master Number Sense Maker.

Andrew Stadel and Daniel Luevanos, for being my co-conspirators. Our exchange of ideas has truly sharpened the edges on this instructional tool. You have been fellow evangelists of this message. May David Lee, Angus, and John Frusciante rock on together forever. Kristen Acosta, Stacy Zagurski and your colleagues at Merlinda Elementary School for your insights into elementary's implementation of Clothesline Math, and to the multitude of other clothesline enthusiasts for blazing a trail through this new territory.

My kindred spirits in the Math Twitter Blogosphere, for making me a phenomenally better teacher. So many of your ideas, like Annie Fetter's Notice and Wonder, have made their way into my teaching, and thus into the examples in this book. #MTBoS rocks!

Diana Kenney, for bringing this book to life so quickly. Conni Medina and all the people at Shell Education who believed so enthusiastically in this project from the very beginning. My editor, Kyleena Harper, for walking me through my first professional publication process, knowing when to nudge and when to kick. Your leadership, organization, and vision, Walter Mladina's photography, and Fabiola Sepulveda's design navigated the uncommon challenges arising from the unique structure of the Clothesline to create a book of which we can all be proud.

My many heroes and mentors, Dr. Tim Kanold, Dr. Uri Treisman, Dr. William Schmidt, Dr. Ilana Horn, Dr. Harris Schultz, Dr. Tom Bennett, Dr. Maggie Smith, Dr. John Star, Dr. Rick Dufour, Dr. Patrick Campbell, Dr. Juli Dixon, Dr. William Doll, Dr. Titu Andreescu, and David Foster, for the solid theoretical foundation on which I have constructed my world view of math education.

My students, for being the reason that I love teaching so much, for playfully receiving the Clothesline in class, and for teaching me far more than I ever taught you.

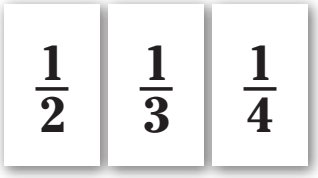
My children, Rylie and Preston, for growing up as the guinea pigs of your math teacher dad. I am so proud of the special adults that you have become. In your own unique way… change the world. My wife, Casie, for being my #1 fan and for teaching me to always teach to the hearts of children as well as to their minds. I love you dearly. As always, thanks for the adventure.

The Lord, for the talents, relationships, knowledge, and opportunities that you have so blessed me with in number and magnitude. May my use of them serve you well.

Dedication

Dedicated to my mother, Jo Lancaster

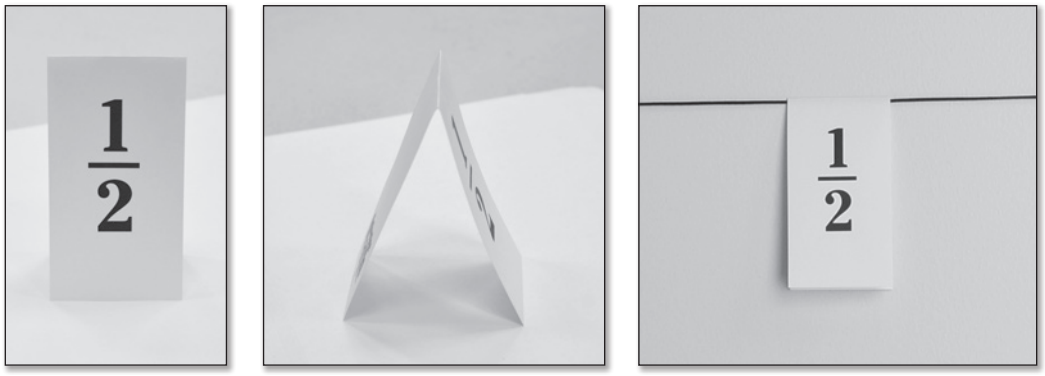
A Universal Example



Objective: Accurately place and space three unit fractions on an open number line.

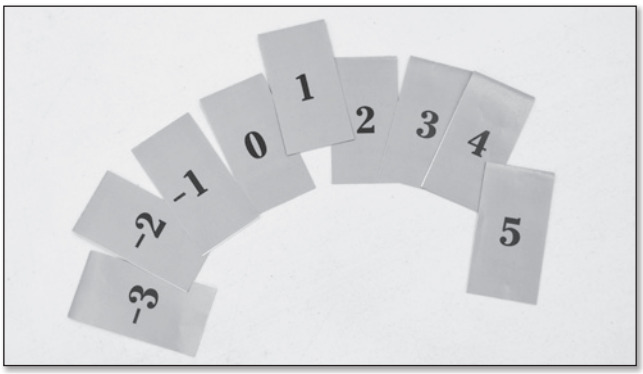
> **Teacher: Good day, class. Today, I am introducing you to the clothesline. This number line will help us better understand the math we are studying. Today, though, we will start with a topic we already know in order to show you how the Clothesline works.**
>
> **First, we call it the Clothesline because it is literally a clothesline with clothespins that we will use to show that certain values are equal.**
>
> **I will give you numbers written on folded cards that look like tents. This shape makes it really easy to hang our numbers on the clothesline.**
>
> *The teacher shows students the folded cards, as shown in Figure I.1.*

Figure I.1 Folded Clothesline Cards



> **You will notice there are no "benchmarks"—those numbers and tick marks you usually see on a number line. That's because this is an open number line, so you get to choose how to label it. When it is your turn to place numbers on the number line, I have benchmarks printed and ready for you on the table at the front of the room.**
>
> *The teacher gestures to the pile of benchmark cards at the front of the room, as shown in Figure I.2.*

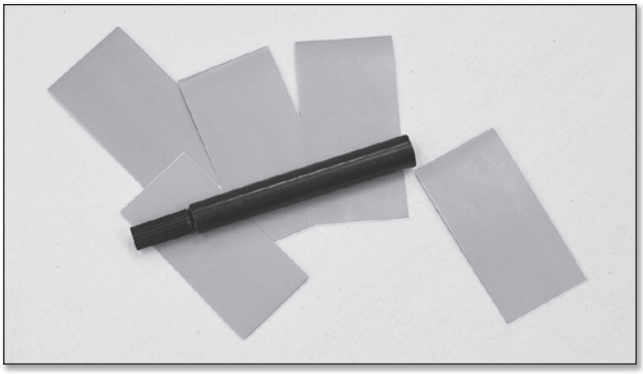
Figure I.2 Sample Benchmark Cards



If you want to use a benchmark that I don't have up here, I have provided blank cards that you may use to create your own.

The teacher shows blank cards sitting next to the benchmark cards, as shown in Figure I.3.

Figure I.3 Blank Cards



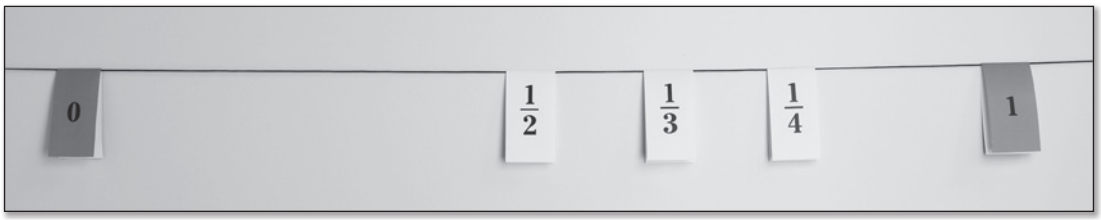
I'm going to give this first set of numbers to one group that will come to the front of the class and place them on the clothesline. While that group is up here, the rest of you will draw your responses on your lapboards with your elbow partners. The person whose birthday is closest to today will go first. Whose birthday is closest?

Steve raises his hand.

Steve. Here is our first set. Class, we are placing $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ on the number line. Please draw a number line on your lapboards with your partners. You are to show any benchmarks you need. Steve, you and your group may use any benchmarks I have provided or none at all. Remember, if you need a number that I did not prepare, you may write it on a blank card.

After some rich conversation at the clothesline, students place the cards in the wrong order (see Figure I.4) and return to their seats.

Figure I.4 First Clothesline Placement



Class, let's see who agrees with this response. Please declare your lapboards.

Student pairs raise their lapboards overhead to display their answers. About half the class agrees with the improper placement of the three values. The other students have them properly placed.

Steve, I see that your group used the benchmarks 0 and 1. So, are you claiming that all three fractions are less than 1, but greater than 0?

Steve: Yes.

Teacher: Why did you place the fractions in this order?

Steve: Because we saw the numbers 2, 3, and 4, so we just counted up.

Teacher: Most of the class agrees with your response here. Does anyone agree with the order of the numbers but for a different reason?

Jackson raises his hand.

Yes, Jackson?

Jackson: I did it in that order because you gave them to us in that order.

Teacher: You disagreed, Kendra. Why?

Kendra: Because $\frac{1}{4}$ is smaller than $\frac{1}{3}$ which is smaller than $\frac{1}{2}$.

Teacher: How do you know?

Kendra: If I cut one pizza into two slices, another pizza into three slices, another into four slices, and then take one slice from each, half of a pizza will be bigger than a fourth of a pizza, and a third will be in the middle.

Teacher: Assuming the pizzas are the same size to begin with?

Kendra: Yes.

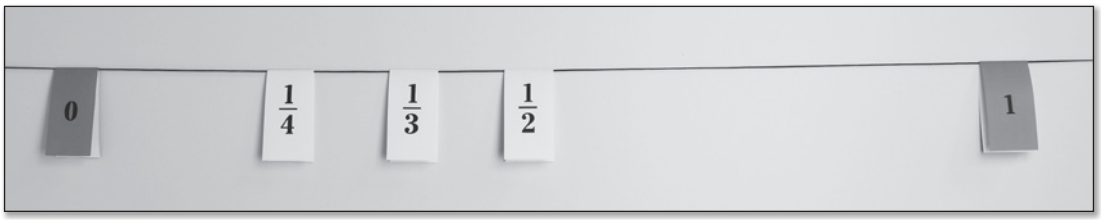
Teacher: You have now heard three different explanations. Let's thumb vote: thumbs-up if you agree with the current placement; thumbs-down if you do not. Are these fractions in the correct order?

The whole class votes thumbs-down.

Steve, your group voted against your own response. If you would like to change it, please feel free to come up and do so.

Steve's group returns to the front of the classroom and rearranges the cards (see Figure I.5).

Figure I.5 Second Clothesline Placement



So, the first thing we want to do with the clothesline is *place* the cards. That means we want to get the values in the correct numerical order. We will talk about spacing in a minute, but do you all agree that these three are now in the right order? All in favor, say "aye."

Class: Aye!

Teacher: I noticed that your group equally spaced the numbers this time. Can someone from the group tell me why?

Akira: Well, we realized $\frac{1}{2}$ had to be in the middle. And $\frac{1}{4}$ is half of a half. Then, $\frac{1}{3}$ goes in between those two.

Teacher: Okay. So after we *place*, we want to *space*. That means we want to slide the values on the number line so they are in their proper locations on the number line. Do you agree these values are properly spaced? Thumb vote.

Most, but not all, vote no.

Teacher: Arturo, why not?

Arturo: $\frac{1}{3}$ needs to be closer to $\frac{1}{4}$.

Teacher: Okay, I'm going to tap $\frac{1}{3}$ in the direction in which you point as a class. When the value is in the right place, clap.

Students point to the left as the teacher moves the cards, as shown in Figure I.6. Several are mumbling, "Keep going." There is finally a collective clap.

Figure I.6 Third Clothesline Placement

Who wants to explain why we stopped here? Cindy.

Cindy: I converted the fractions to decimals.

Teacher: How did that help you?

Cindy: You get 0.25, 0.33, and 0.50. I know that 33 is closer to 25 than 50, so I kind of estimated where 0.33 would go.

Teacher: Thank you. Did anyone do it a different way? Skylar.

Skylar: Common denominators give you $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{6}{12}$. The number 4 is closer to 3 than 6. So, I estimated by counting 3, 4, 5, 6.

Teacher: Thank you. Any other ways to do this? Jesse.

Jesse: On our board, I used my pencil to measure $\frac{1}{3}$ and then checked whether that length fit into 1 three times.

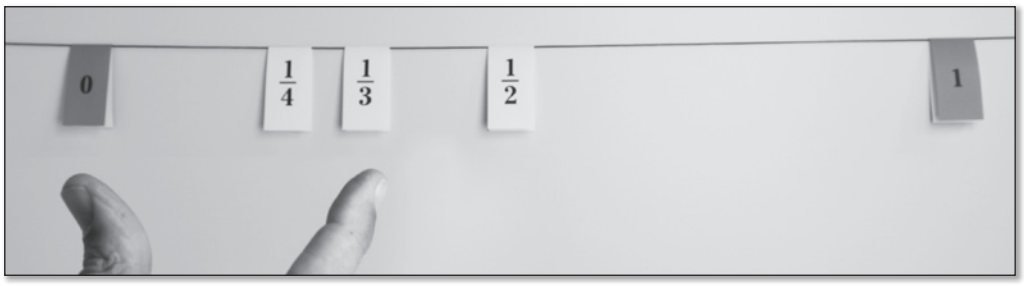
Teacher: So, you are using something we call finger reasoning. Everyone, close one eye and place your fingers in front of you so you can remotely measure from 0 to $\frac{1}{3}$.

Students close their eyes and hold up their fingers.

Now, see whether three times that distance is equivalent to 1.

Students use their fingers to remotely measure the spacing of the fractions on the clothesline, as shown in Figure I.7.

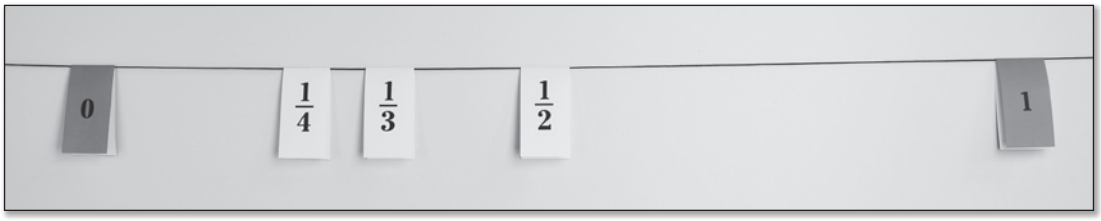
Figure I.7 Sample Finger Measuring



If you now agree with both the placing and the spacing of our three values, clap twice.

The class gives a resounding two claps. The cards stay as they are, as shown in Figure I.8.

Figure I.8 Final Clothesline Placement



Now class, it is time to record our discussions, deductions, and decisions on your activity sheets. On the three blanks, write $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Then, record them on the number line provided with the benchmarks. Be sure to be just as diligent about placing and spacing on your paper as we were on our clothesline. Well done, math crew.

123 Why These Numbers?

The key to a good clothesline lesson is in the selection of the proper values to pose to the class. For example, I could have chosen any three fractions, but $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are the most common. In addition, the consecutive denominators of 2, 3, and 4 keep calculations simple and bring out the misconceptions that a larger denominator implies a larger fraction.

The Key Questions

- Why did you include 0 and 1 as benchmarks?
- Why did you not include any benchmarks?
- Do we need benchmarks?
- Why did you place them in that order?
- How do you know $\frac{1}{4}$ is smaller than $\frac{1}{3}$?
- Did anyone come to the same conclusion, but in a different way?
- Why do you disagree?

Analysis

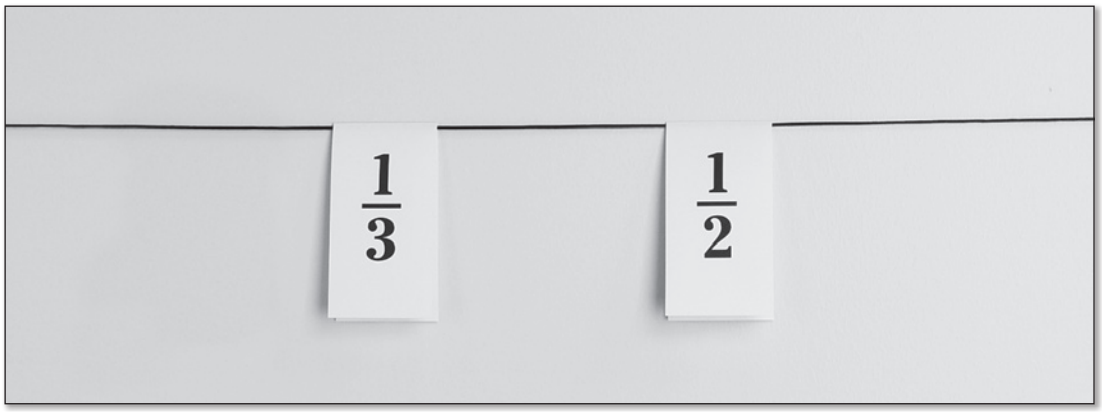
For the record, this example occurred during an Algebra 2 class. A room full of juniors in high school struggled with accurately placing the three simplest fractions on a number line. With these three fractions, I anticipated some students would make the error of placing them in ascending order of the denominator, as some indeed did. I did not expect students to simply copy the order in which I offered the three values. This reveals just how trained students are to copy from and regurgitate back to the teacher.

The strategies of classmates, particularly those of decimal conversions and common denominators, helped many students sort their thinking on fractions. The proportional reasoning encouraged by the finger reasoning was beneficial to all and will be further enhanced as we revisit number sense on the clothesline throughout the school year.

Extension

Place a fraction on the number line that is between $\frac{1}{3}$ and $\frac{1}{2}$.

Figure I.9 Final Clothesline Placement



The Structure of This Book

What you just read was a typical introduction to Clothesline Math.

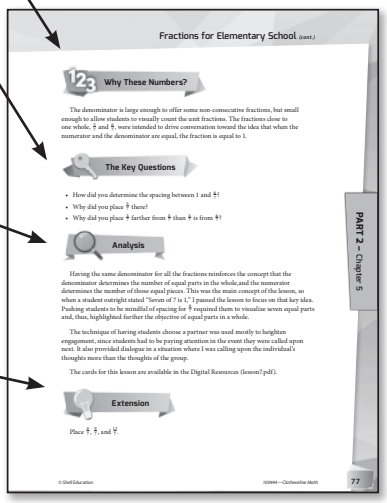
The majority of this book is filled with lessons like these. The values or expressions to be placed on the number line are posed first, followed by a mock classroom discourse that leads to the correct placement on the clothesline, which is to be recorded on the *Clothesline Math* activity sheet (pages 315–316) at the end of each lesson. A teacher debrief finishes each model lesson with the following format:

Why These Numbers: This is an explanation of the importance behind the choice of numbers for each lesson.

Key Questions: Clothesline Math is intended for discourse, not lecture. Since good classroom discussion is led by questions, not statements, a bank of prompts is offered for each lesson.

Analysis: This is the reflection on the lesson vignette, including discussion of student thinking and teacher decisions during the lesson.

Extension: This is an additional activity suggestion to be used for differentiation, assessment, or independent assignment.


Fractions for Elementary School *(cont.)*

Why These Numbers?

The denominator is large enough to offer some non-consecutive fractions, but small enough to allow students to visually count the unit fractions. The fractions close to one whole, $\frac{6}{7}$ and $\frac{8}{9}$, were intended to drive conversation toward the idea that when the numerator and the denominator are equal, the fraction is equal to 1.

The Key Questions

- How did you determine the spacing between 1 and $\frac{6}{7}$?
- Why did you place $\frac{8}{9}$ there?
- Why did you place $\frac{4}{5}$ farther from $\frac{6}{7}$ than $\frac{8}{9}$ is from $\frac{6}{7}$?

Analysis

Having the same denominator for all the fractions reinforces the concept that the denominator determines the number of equal parts in the whole,and the numerator determines the number of those equal pieces. This was the main concept of the lesson, so when a student outright stated "Seven of 7 is 1," I paused the lesson to focus on that key idea. Pushing students to be mindful of spacing for $\frac{8}{9}$ required them to visualize seven equal parts and, thus, highlighted further the objective of equal parts in a whole.

The technique of having students choose a partner was used mostly to heighten engagement, since students had to be paying attention in the event they were called upon next. It also provided dialogue in a situation where I was calling upon the individual's thoughts more than the thoughts of the group.

The cards for this lesson are available in the Digital Resources (lesson7.pdf).

Extension

Place $\frac{4}{5}$, $\frac{3}{7}$, and $\frac{7}{8}$.

PART 2 – Chapter 5

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This text also includes justification for teaching number sense at all grade levels, the simple materials and setup needed to conduct clothesline lessons in your classroom, and techniques for effectively facilitating the discourse spawned by our favorite number line tool. With these examples and tips, I hope to empower your teaching with Clothesline Math.

Teaching with the Clothesline

Chapter 1

The Master Number Sense Maker

In February 2015, I attended the annual MaTHink Conference in Riverside, California. I sat in on a session that a friend of mine, Kelli Wise, was presenting on Clothesline Math. Although I had seen the clothesline used for fractions and decimals, I was intrigued by her description of demonstrating algebra on an open number line. As promised, Wise placed a variable on the clothesline, and my mind was blown. As she continued through her examples, the clothesline touted its prowess of promoting conceptual understanding of algebra, revealing students' misconceptions of variables, strengthening basic number sense, enhancing critical thinking, and generating mathematical discourse. I was in awe of this display of instructional muscle. Wise concluded her session by offering tips on choosing values to place on the number line and on asking the right questions to lead discourse, which I had already learned from her skillful presentation.

At the end of the session, my friend sitting next to me, Tim McCaffrey, claimed, "The clothesline is the master number sense maker." I wholeheartedly agreed and returned to my classroom excited to implement this new tool with my own Algebra students. The Clothesline proved to be even more powerful in the classroom since students didn't know all the answers as did the room full of math teachers at the conference. I started to see more potential for the clothesline as I experimented with it in my geometry classes, developing new applications and improving my own skills for its facilitation.

I began to demonstrate it for other teachers as part of my math coaching duties, as well as at district trainings and math conferences, and the popularity of clothesline grew. To assist in its expanding use, I created the website www.clotheslinemath.com and started the Twitter hashtag #clotheslinemath. With more teachers at various grade levels in various math courses sharing ways of using the Clothesline, enthusiasm for it swelled. I started receiving thank you emails and photographs of Clothesline lessons from across the country. Then, I went to the annual conference for the National Council of Teachers of Mathematics in 2017 and saw five different sessions besides my own in which the Clothesline was being presented. I realized that my love for the "Master Number Sense Maker" was shared by many.

The Need for Number Sense

The clothesline is more than a trending activity, however. It serves a crucial, yet overlooked, purpose in mathematics education—the need to teach number sense.

Several longitudinal studies show that students who lack number sense in primary grades are nearly guaranteed to fail their high school math courses, and students who have strong number sense are nearly guaranteed to excel in their high school math courses (Jordan 2010; Duncan 2007). Number sense has a similar influence on students' success in math as reading has on their school success overall. In fact, "mathematics difficulties and disabilities have their roots in weak number sense" (Jordan 2010, 2).

So, what is number sense exactly? In essence, it is a "child's fluidity and flexibility with numbers" (Gersten and Chard 1999, 3). In other words, being fluent in math is much the same as being fluent in a language. To be fluent in a language, one must know the fundamentals of the language, but one must also have the ability to respond to a question or take part in a conversation with sentences that have not been simply memorized and regurgitated. Just as people need to be fluid and flexible with their words, students must be able to respond in the same fluid and flexible way when encountering unique math problems. Their ability to do this relies heavily on their number sense.

Number sense is acquired through exploration and play. Much of the number sense children arrive with in kindergarten is acquired informally in the home in much the same way their language skills are acquired. Students who are more engaged with words at home enter school with larger vocabularies and stronger language structures. Thus, they are better prepared to learn how to read and write. Students who are engaged with numbers and their properties at home enter school better prepared to learn math, too. However, students do not obtain language readiness by studying pre-K vocabulary lists. They informally interact with their language, usually through conversation with adults. Correspondingly, students do not obtain number sense by memorizing lists of math facts. Instead, they interact informally with numbers. Oftentimes this comes through games, such as card and board games, in which students are required to count, add, and strategize. Duncan (2007) describes that "play-based, as opposed to 'drill-and-practice,' curricula designed with the developmental needs of children in mind can foster the development of academic and attention skills in ways that are engaging and fun" (8). The clothesline straddles that boundary between developing informal play and the formal needs of classroom management. This is enormously helpful when teaching students who do not have adequate number sense to learn the content at their current grade level.

Teaching Number Sense and Prerequisite Skills

Why, then, does the building of number sense get so little attention in the mathematics classroom? There are two valid reasons. The first is the lack of time. Teachers are already hard pressed to cover current content standards. The second is the lack of resources to help teach number sense. Worksheets for procedural practice fall short of building number sense, take too much time, and often have already failed students in their prior experiences. Therefore, teachers need a simple tool that can quickly strengthen students' prior knowledge

while simultaneously moving them forward with grade-level content. The open number line serves as such an instrument that can be used at the beginning of a unit, as a warm-up, on assignments, or as part of assessments.

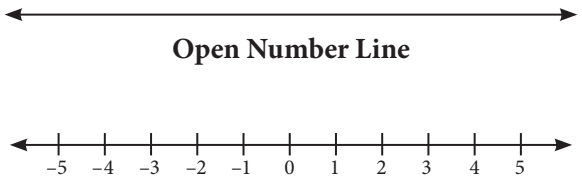
Besides a lack of number sense, there is another hindrance to learning math: the lack of prerequisite skills. This is well-known to most math teachers. At all grade levels, one of the most common complaints made by teachers is that students are not prepared for the current grade level or math course. Calculus teachers claim students don't flunk calculus, they flunk algebra. Students don't flunk algebra, they flunk middle-school math. If they can't multiply negative numbers, it's because they can't multiply positive numbers. Elementary school students can't add two-digit numbers because they can't add single-digit numbers. Nearly every teacher agrees: Students don't flunk current content; they flunk prior content.

So, if we are all in consensus that number sense is a critical ability too many students lack and that many students are missing several of the skills required to be successful, then the obvious response from all those teaching math should be to teach number sense and the prerequisite skills. The Clothesline is prepared for the task.

The Open Number Line

As shown in Figure 1.1, an open number line is one that does not have benchmarks already established. In other words, the numbers 0, 1, 2, 3, etc. that you would see on a traditional number line are missing.

Figure 1.1 Open versus Traditional Number Line



Open Number Line

Traditional Number Line

A lack of benchmark numbers requires students to establish a scale. Demanding this of students serves two purposes. The first is to help students build understanding of cardinality, a sense of magnitude, and proportional reasoning. The second is that the new twenty-first century standards insist that high school students be able to "choose and interpret the scale and the origin in graphs and data displays" (CCSS HSN.Q.A.1) and students of all levels be capable at "considering the units involved; attending to the meaning of quantities, not just how to compute them" (CCSS MP 2). There are standards that explicitly claim the use of number lines throughout the elementary and middle school grades, such as "represent a fraction $\frac{a}{b}$ on a number line diagram by marking off lengths $\frac{1}{b}$ from 0," (CCSS 3.NF.A.2.A) and "Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about…double number line diagrams" (CCSS 6/RPA.3).

The open number line is a means and an end. It is a tool to develop number sense and reteach prerequisite skills, but having students proficiently use it is a goal in and of itself.

NCTM's Eight Mathematics Teaching Practices

The National Council of Teachers of Mathematics (2014) laid out "eight effective teaching practices" in their publication *Principles to Actions: Ensuring Mathematical Success for All.* This well-regarded document claims that effective mathematics teachers do the following:

1. Establish mathematics goals to focus learning.
2. Implement tasks that promote reasoning and problem solving.
3. Use and connect mathematical representations.
4. Facilitate meaningful mathematical discourse.
5. Pose purposeful questions.
6. Build procedural fluency from conceptual understanding.
7. Support productive struggle in learning mathematics.
8. Elicit and use evidence of student thinking.

The Clothesline helps implement all of these, particularly facilitating math discourse, building conceptual understanding, and eliciting evidence of student thinking. In other words, the clothesline can aid you in becoming a more effective twenty-first century math teacher. As you read the lessons in this book, notice how the card sets are chosen and placed to drive students toward cognitive dissonance. I joke to my pupils that I purposely try to get their brain cells to fight. This intellectual challenge is what brings conceptual understanding to the forefront, so students gain procedural fluency more easily.

The lesson vignettes in this book also serve as extensive models of how to facilitate mathematical discourse. They demonstrate solid teacher questioning and guidance of the discussion as well as typical statements made by students. The entire conversation, of course, is driven by the evidence of students' reasoning found in their responses on the Clothesline. The ultimate goal of these lessons is to develop the mathematical understanding of the students. NCTM calls for teachers to take students beyond simple answer getting. The Clothesline answers that call.

The Eight Standards of Mathematical Practice

The new twenty-first century standards not only have framed what students should know but also how they should think. These "habits of mind" are most commonly known as Standards of Mathematical Practice. However, in some states they are also known as Mathematical Processes. They are all very similar and nearly identically described. This noble goal of teaching students to think is the hallmark of the new standards, and thus is the hardest part of the new standards to teach students. They are not teacher moves like NCTM's Practices; they are student habits to be instilled at every grade level. They are exquisitely stated as the following:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The Clothesline can be used to teach and reinforce any one of these. However, I think its strengths lie in supporting Practices 1, 2, 3, 6, and 7:

- **Make sense of problems and persevere in solving them—**Sense making is one of the key features of the clothesline. Taking symbols and numbers from paper to the number line forces students to grapple with the relationships and properties of these values until they are understood, not just memorized.
- **Reason abstractly and quantitatively—**Reasoning quantitatively is a given…it's a number line after all. Of course it's going to get students to think about numbers! But instead of students just seeing a value written as 0.003, placing it very close to zero on the clothesline helps them understand the relative minuteness of this quantity.

  Additionally, reasoning abstractly is the holy grail of an education. While concrete reasoning involves things we can see and touch, abstract reasoning involves things that we cannot see and touch. It means we think about ideas. For example, $\frac{1}{2^1}$ and $\frac{1}{2^2}$ are easy to think about, because I have experience cutting something in half, and I have taken half of a half before. But what about $\frac{1}{2^{100}}$? I have never repeatedly cut something in half 100 times. Therefore, placing this fraction on the number line requires abstract reasoning. Many students place it well to the right of the number line, claiming that since the exponent is so big the value is close to infinity. Others place it well to the left because they figure if something gets smaller 100 times, it must approach negative infinity. Imagining cutting something in half 100 times (without actually doing the cutting) and realizing the size of the quantity is approaching 0 takes some serious abstract thinking.

- **Construct viable arguments and critique the reasoning of others—**This is the fun part of the clothesline. Getting students to debate ideas in math leads them to a deeper understanding of mathematics, but even more so, it trains them to think like mathematicians. Having the confidence to defend their own conjectures and challenge others is a phenomenal gift that Clothesline Math can give our students.
- **Attend to precision—**Attending to precision is given particular importance when using proportional reasoning to get the spacing of the cards correct and in the promotion of properly using academic vocabulary.
- **Look for and make use of structure—**This process is often required to accurately respond to Clothesline prompts. Several structures were being wrestled with by the students in the Introduction example: the denominator establishes the total parts of the same whole, fewer parts of the same whole implies larger parts, and the parts are equivalent. These are some of the underlying structures of unit fractions that become more evident on a number line than when simply writing them with pencil and paper.

My final thoughts on the relationship between teaching mathematical practices and processes and the Clothesline deal with dual objectives. Every lesson should have some type of instructional objective. A twenty-first century math lesson should always have two: one content objective and one practice/process objective. I do this in my classroom on a regular basis. For example, "We will *persevere* in solving problems involving *trigonometric ratios*" is a dual objective I have used in my geometry class. While the content standard is focused on applying trigonometry, the practice standard is on persevering. I state this at the beginning and revisit the idea of not quitting throughout the lesson as students work on higher level trigonometry problems.

Since Clothesline activities are rarely used as stand-alone lessons, I have not stated dual objectives here in this book; I have stated only the content objective of the lesson. Besides, nearly all of the math tasks I have used in my own classes can be used for teaching several mathematical practices/processes, even though we only want to focus students on one for any given lesson. For these reasons, I leave it up to you as to which practice/process you want to emphasize in any given clothesline lesson. I strongly encourage you, though, to make it a point to instill these habits of mind on a daily basis.

Chapter 2

Setting Up Your Clothesline

Now that I have made the case for why the Clothesline is such a powerful tool for teaching mathematics, let's dive into what you will need to make it happen in your classroom. The following sections will outline the items you will need as well as how to utilize each element in your lessons.

The Materials

The Clothesline gets its name from being, well, literally a clothesline. You can purchase a clothesline and a package of clothespins for less than $5. You will also need a few other materials for attaching the clothesline to the wall and for students to participate at their desks. In this chapter, I will discuss several options for you to choose from, but the materials needed to conduct an amazing Clothesline lesson are minimal. For example, the materials list for my own classroom is as follows:

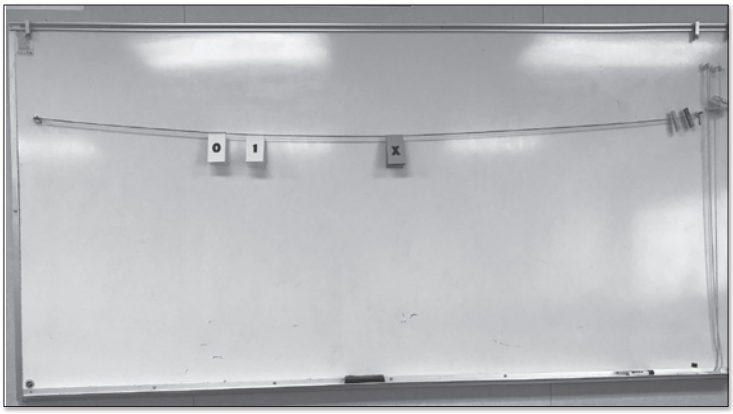
- 2 nylon clotheslines
- 12 clothespins (assorted colors)
- 4 magnetic hooks (2 per clothesline to attach to the whiteboard)
- clothesline cards (number cards, benchmark cards, and blank cards)
- 5 oversized envelopes (to store the cards)
- lapboards (1 board per student pair or group)
- *Clothesline Math* activity sheet (pages 315–316)

Let me show you the setup I use in my classroom as well as examples I have seen from other classrooms.

The Front of the Class

In my classroom, I use magnetic hooks on my whiteboard (see Figure 2.1). The clotheslines are attached and hang nicely to the side, so they are always ready to go. The added bonus of the whiteboard is that I can write notes and examples above and below the clothesline.

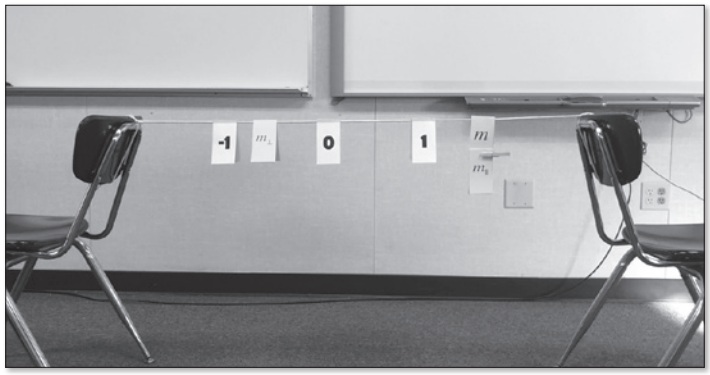
Figure 2.1 Sample Clothesline on Whiteboard



This is not the only setup, however. The first time I saw the Clothesline demonstrated, it was suspended across the corner of a room (see Figure 2.2), so that is how I initially did it. I have seen clotheslines tied between individual chairs (see Figure 2.3), stacks of chairs, and even homemade PVC posts. I once made a makeshift clothesline with a computer printer cable and two music stands. I've even seen a hi-tech version on an interactive whiteboard.

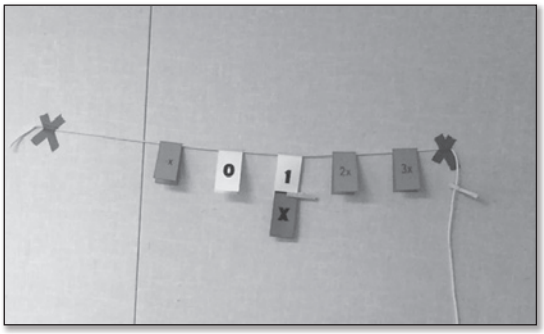
Figure 2.2 Corner Clothesline

Figure 2.3 Chair Clothesline



The easiest setup, of course, is a clothesline pinned or taped to a wall (see Figure 2.4).

Figure 2.4 Wall Clothesline



Choose what works best for you and your classroom. All that matters is that you get the open number line in front of students.

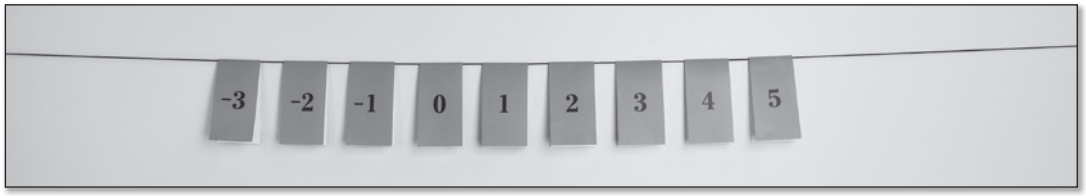
The Cards

Clothesline cards are best folded in half so they hang nicely on the clothesline. This also allows them to easily move back and forth along the line. I typically cut three cards from one sheet of paper. Cards that have longer expressions may only fit two on a single sheet. While you can handwrite them, common number cards for the lessons in this book have been included in Appendix C (pages 317–335). All the cards needed for each lesson are available as individual PDFs in the Digital Resources. You can also find a multitude of prepared cards online at www.clotheslinemath.com.

Digital resources to support the lessons in this book are available online. A complete list of available documents is provided on page 336. To access the digital resources, go to this website: http://www.tcmpub.com/download-files. Enter this code: 34583417. Follow the on-screen directions.

While the open number line offers few to no benchmarks to start, benchmarks should be available to students. Benchmark cards need to be printed and available, usually sitting on a desk or table near the front of the room. As shown in Figure 2.5, benchmark cards typically have whole numbers from 0 to 10 for elementary school and integers from -3 to 5 for secondary.

Figure 2.5 Integer Benchmark Cards



For courses that teach angle measure, you will also want benchmarks for the most common degree measures of 0°, 30°, 45°, 60°, 90°, 180° (see Figure 2.6).

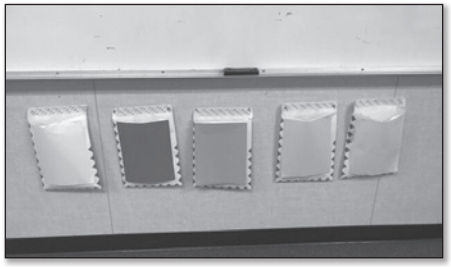
Figure 2.6 Degree Benchmark Cards

You should also have several blank cards on a table nearby for students to use.

Like the fraction example in the Introduction, a lesson's card set is usually comprised of three values. However, some lessons may vary. I store my card sets in oversized envelopes that I keep pinned to the front wall just under the board where the clothesline is displayed. I also color-code my cards, though this is not at all necessary. As shown in Figure 2.7, I print the cards on colored paper as follows:

- **White:** benchmarks
- **Green:** numbers
- **Red:** single variables and statistic symbols
- **Orange:** algebraic expressions
- **Yellow:** angle measures and drawings

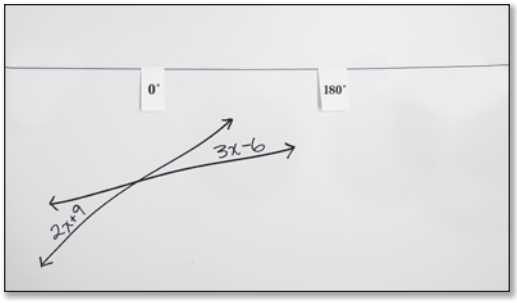
Figure 2.7 Color-Coded Card Sets

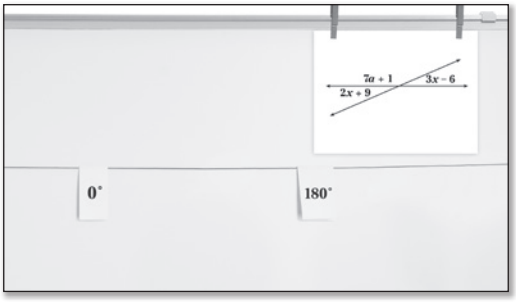


For readability purposes of the photos in this book, all number cards will be on white paper, and benchmarks will be on colored paper.

Drawings are often needed for Clothesline lessons on geometry and functions. While I often have these drawings on paper, I also display drawings by enlarging them on the whiteboard or interactive whiteboard for the sake of easier viewing (see Figure 2.8).

Figure 2.8 Sample Drawings on Whiteboard





The Rest of the Class

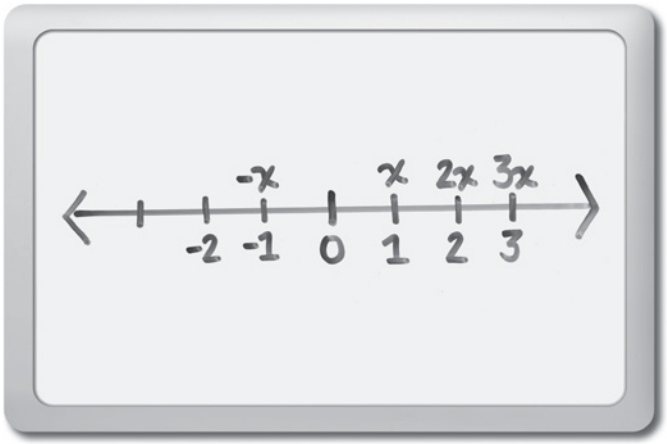
It is very important that, while one group is at the clothesline, the rest of the class has something to do at their desks to engage them in the activity. I learned this the hard way.

One day, I was telling a fellow teacher how excited I was to try the clothesline for the first time. He asked if he could come watch. My first thought was, "Did you not hear me say that I was doing this for the first time?" As a fellow teacher, he should have known that we teachers don't want people watching us until we have the lesson down pat. However, I was a math coach for the department and needed to promote transparency and risk taking, so I had no recourse but to invite my buddy along.

I introduced the Clothesline to the class and sent a group of students up to place the cards. The conversation within the group was amazingly rich. But, it was also long. Do you want to guess what happens when one group of students stands with their backs to the class for an extended period of time and you haven't given the rest of the class something to do? You've got it…the animals escaped from the zoo. And, my colleague watched the whole thing implode.

We debriefed afterward and both agreed the group at the front of the class proved how engaging and instructional the Clothesline is, but the other students needed something to do at their desks to engage them as well. The next day, he came back to observe another Clothesline activity again. This time, we had students pair up and use lapboards (see Figure 2.9) to give their responses to the prompt being discussed at the front of the room. This not only helped classroom management and engaged all students in the mathematics of the activity, it also greatly enhanced classroom discourse. We were able to discuss who agreed and disagreed with the card placement on the clotheslines, plus students had the opportunity for the same rich discussion with their partners that the group at the clothesline had.

Figure 2.9 Sample Lapboard

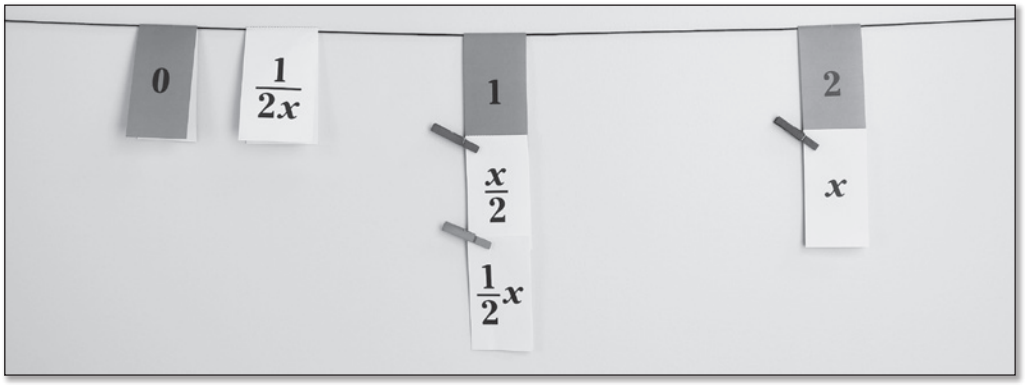


I find lapboards to be the easiest tool to engage students at their desks but not the cheapest. If you do not have a set of lapboards available and your school does not have the funds to purchase a class set for you, you can make them by inserting blank paper or graph paper into sheet protectors.

The Clothespins

When students place equal values on the same location of the clothesline for the first time, they inevitably set one atop the other, so that one of the values is no longer visible. I joke with them at this point, "Hey, where did the other one go?" When they reveal the other value underneath, I ask, "Why did you do that?" Once they accurately claim the values are equal, I grab a clothespin and pin the values together vertically so both are visible (see Figure 2.10). The clothespins are used to show equivalency. Sometimes, there may be more than one pair or more than two values that are equivalent to one another.

Figure 2.10 Equivalent Cards Pinned



Sometimes, equivalency becomes the focal point of the lesson. That is the time to make the move to multiple clotheslines.

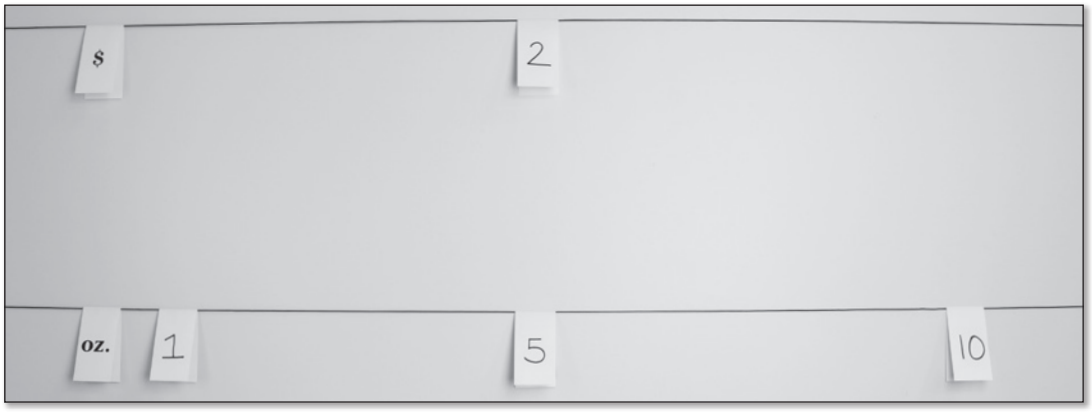
The Double and Triple Clothesline

A double clothesline is an effective setup when you have situations with different units of measure, like with rates, or when teaching multiple forms of a value, such as percentages and decimals. For example, a double clothesline is a terrific tool for presenting the concept of unit rate.

Consider this scenario, shown to me by Kristen Bennett of Tustin USD: Jody buys a 5-ounce serving of frozen yogurt for $2. Her older brother, Timmy, buys the 10-ounce serving. How much does Timmy's yogurt cost? What is the price of yogurt per ounce?

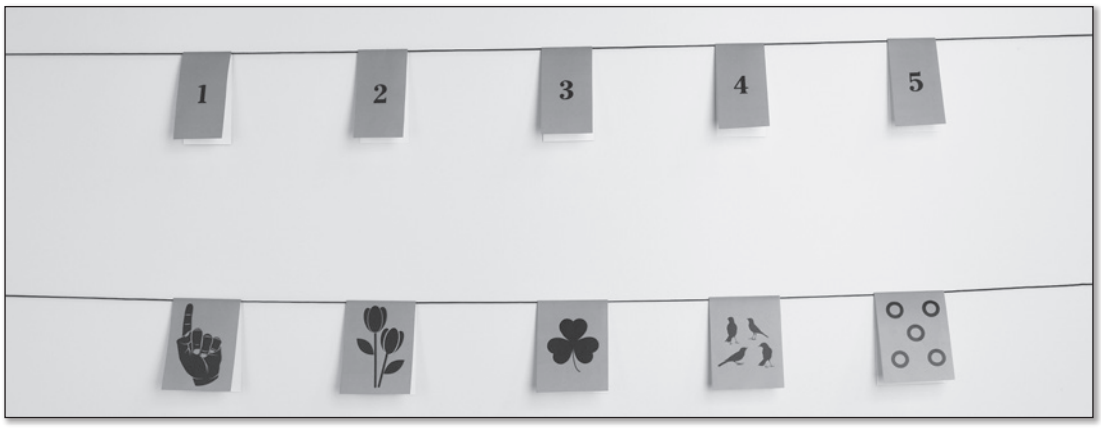
In this example, pinning a 5 and 2 together on one clothesline would be confusing and inaccurate. We are not claiming that 5 is equal to 2! We are claiming there is a relationship between ounces and dollars. As shown in Figure 2.11, the double clothesline allows us to label these units in a way that illuminates the unit rate concept of dollars per ounce.

Figure 2.11 Sample Double Clothesline—Price versus Ounces



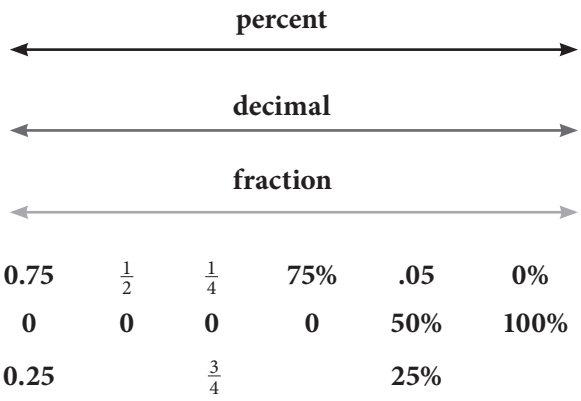
This way of comparing different units or representations with multiple clotheslines spans all grade levels, as you can see in Figure 2.12.

Figure 2.12 Sample Double Clothesline—Number versus Pictoral



I have even seen a triple clothesline tie together the various forms of numbers (see Figure 2.13)

Figure 2.13 Sample Triple Clothesline



The Activity Sheet

Students can easily record the results of the Clothesline lessons on notebook paper. However, the *Clothesline Math* activity sheet (pages 315–316) is available in Appendix C. Its use is discussed in Chapter 3.

Name: _____ Date: _____

Clothesline Math

Directions: For each set, record the given values, expressions, or drawings. After the discussion of their placement on the clothesline, record them on the number line.

1. _____, _____, _____

2. _____, _____, _____

3. _____, _____, _____

Chapter 3

Making the Most of Your Clothesline

Now, you have what is needed to bring Clothesline Math to your classroom. The next step is to review how you can best facilitate it.

Choosing Values for the Card Sets

The key to an effective Clothesline lesson is choosing the appropriate values for the card set. This set of values needs to create some kind of cognitive dissonance that will assist you in illuminating the mathematical objective. For example, math teachers know that students have a very difficult time discerning between $(-3)^2$ and -3^2, yet this is a critical concept for evaluating expressions. One of the best ways to clear up any misconception is to take it to the Clothesline. Therefore, the card set for this lesson would be the expressions $(-3)^2$, $-(3)^2$, and -3^2. The last two terms are equivalent since both equal negative 9, but the first equals positive 9.

Choosing values in which some are equal and others are not, while still looking very similar, adds power to a lesson's ability to clarify the mathematical concept that is confusing students. Card sets for the Clothesline lessons in this book are available in the Digital Resources. In the event you want to create your own card sets, take your time choosing values that you will use, making sure to choose values that best drive student attention toward the point of proper understanding.

Questioning with Purpose

One of NCTM's Eight Teaching Practices—purposeful questioning—is an especially important teacher move during Clothesline lessons. You need to do more than simply state, "That's wrong. You should have placed it here." Good questioning drives class discussions to the core mathematical objectives. Therefore, two things are offered in this book to support your purposeful questioning. The first are the vignettes, which model questioning by the teacher. You will notice the teacher often responds to student answers with another question that drives discussion closer to the lesson objective. Sometimes, the purpose of a question is just to have the next student restate the previous student's response to keep the class actively engaged. Other times, the question is intended to reveal multiple solutions or strategies.

The second tool to support purposeful questioning is a set of suggested questions for each lesson. Eventually, you will get comfortable generating your own questions. In the meantime, these will help you get started and will show just how far you can push student thinking with the Clothesline. To help in this endeavor, let us explore a proven framework for leading classroom discourse in mathematics.